The reactant concentration spectrum in turbulent mixing with a first-order reaction

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The power spectrum of a passive scalar contaminant undergoing a first-order chemical reaction and isotropic turbulent mixing is deduced for three different spectral ranges: (i) the inertial-convective range; (ii) the viscous-convective and viscous-diffusive ranges for very large Schmidt number; (iii) the inertial-diffusive range for very small Schmidt number. The analysis is restricted to stationary, locally isotropic fields, and to systems so dilute that the heat of reaction has no effect on the reaction rate.

Introduction

A clarification and extension of the stationary, locally isotropic, turbulent mixing problem has recently been given by Batchelor (1959), and Batchelor, Howells & Townsend (1959). After preserving the Obukhov (1949)–Corrsin (1951) spectral result,

$$G(k) \sim \epsilon_\theta \epsilon^{-\frac{1}{3}} k^{-\frac{5}{3}}, \tag{1}$$

with proportionality constant of order one, for the inertial-convective wave-number range, Batchelor focused attention on fluid + contaminant fields with very large 'Schmidt number' or 'Prandtl number', $\nu/\eta \gg 1$. ν is the kinematic viscosity, η is the diffusivity, k is the wave-number, ϵ is the turbulent energy-dissipation rate, ϵ_θ is the corresponding scalar contaminant field property, the rate of destruction of $\overline{\theta^2}$ by diffusion.

He pointed out that for $\nu/\eta \gg 1$ and large Reynolds number, there will be a viscous-convective wave-number range,

$$\left(\frac{\epsilon}{\nu^3}\right)^{\frac{1}{4}} \ll k \ll \left(\frac{\epsilon}{\nu\eta^2}\right)^{\frac{1}{4}}, \tag{2}$$

in which the velocity spectrum is strongly affected by viscous forces, while the scalar field spectral transfer is predominantly convective. His analysis for $k \gg (\epsilon/\nu^3)^{\frac{1}{4}}$ yielded

$$G(k) \approx 2\epsilon_\theta \frac{\nu^{\frac{1}{2}}}{\epsilon^{\frac{1}{2}}} \frac{1}{k} \exp\left\{-2\left(\frac{k}{k_B}\right)^2\right\}, \tag{3}$$

where $k_B \equiv (\epsilon/\nu\eta^2)^{\frac{1}{4}}$. This applies from the viscous-convective range (where the exponential is of order unity) all the way to $k \rightarrow \infty$.

Then Batchelor, Howells & Townsend considered the opposite extreme of a fluid-contaminant combination for which $\nu/\eta \ll 1$, in which there can exist an inertial-diffusive spectra range

$$\left(\frac{\epsilon}{\eta^3}\right)^{\frac{1}{4}} \ll k \ll \left(\frac{\epsilon}{\nu^3}\right)^{\frac{1}{4}}. \tag{4}$$

Here the turbulence spectrum is of Kolmogorov type

$$F(k) \sim \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}, \tag{5}$$

with proportionality constant of order one, while the contaminant spectrum is strongly diffusive. For this range they deduced essentially that

$$G(k) \approx \frac{1}{3} \frac{\epsilon_\theta \epsilon^{\frac{2}{3}}}{\eta^3} k^{-\frac{17}{3}}. \tag{6}$$

The present study is aimed at generalizing equations (1), (3) and (6) to include the effect of a first-order chemical reaction, assuming a system so dilute that associated temperature changes have negligible effects on the reaction rate and flow. Some preliminary results on decaying, reacting fields have been discussed (Corrsin 1958), and the formal consequences of the quasi-Gaussian hypothesis have been analysed (O'Brien 1960).

The differential equation for concentration of a dilute contaminant undergoing a first-order chemical reaction along with the convection and molecular diffusion is

$$\frac{\partial \theta}{\partial t} + u_i \frac{\partial \theta}{\partial x_i} = \eta \nabla^2 \theta - C\theta, \tag{7}$$

where C is a constant. With less dilute reactant, it will be necessary to include the heat of reaction and the temperature dependence of C. $\mathbf{u}$ is the turbulent velocity field, η is the (constant) diffusivity, t is time, $\mathbf{x}$ is the co-ordinate vector. As pointed out earlier (Corrsin 1958; O'Brien 1960), this particular reaction-rate term is not spectrally selective, so in a freely decaying field it merely multiplies the mixing wave-number spectrum by an additional (exponential) decay in time.

For stationary states, however, such simple arguments cannot be applied; we must pursue spectral elements in time as they migrate to ever larger wave numbers in the stationary spectrum. For the inertial range analysis of the turbulence itself this is essentially the approach used rather informally by Onsager (1949), arriving independently at Kolmogorov's result $F \sim k^{-\frac{5}{3}}$. For the viscous-convective spectral range in scalar mixing, this method of reasoning was introduced more rigorously by Batchelor (1959). Direct dimensional reasoning will not yield the explicit result of equation (23) because the reaction introduces an additional dimensional constant without increasing the number of dimensions.

This paper is restricted to local isotropy for both velocity and reactant fields.

Pure mixing in the inertial-convective range

As a prelude to the following section, it is instructive to re-derive the k-behaviour of (1) by a method like Onsager's. We restrict our consideration to a k-range in which the transfer of $\overline{\theta^2}$ spectral elements is purely convective. This means that

$$k \ll \left(\frac{\epsilon}{\nu \eta^2}\right)^{\frac{1}{4}}, \tag{8}$$

if $\nu/\eta \gg 1$, and that

$$k \ll \left(\frac{\epsilon}{\eta^3}\right)^{\frac{1}{4}}, \tag{9}$$

if $\upsilon/\eta \ll 1$.

Employing the crude concept of a spectral 'energy' cascade in which the $\overline{\theta^2}$ spectral elements 'jump' to successively larger k, we introduce the following assumptions.

(*a*) The cascade wave-number sequence is a geometrical progression, so that

$$\Delta k \approx k. \tag{10}$$

For turbulence dynamics this is suggested by the non-linearity of the Navier–Stokes equations; here we appeal to the quasi-non-linear traits of (7), due to the convective 'forcing function' coefficient $\mathbf{u}$.

(*b*) The part of $\overline{\theta^2}$ transferred in a jump at wave-number k is $G(k)\,\Delta k$. We take it to be $\approx G(k)\,k$.

(*c*) The characteristic time for each jump depends, of course, on the velocity spectrum $F(k)$, and the simplest dimensional possibility is

$$\tau(k) = k^{-\frac{3}{2}}F^{-\frac{1}{2}}. \tag{11}$$

The use of this form for the spectrally local convection time restricts us to the inertial part of the turbulence spectrum.

In the inertial-convective range, the rate of flux of $\overline{\theta^2}$ spectral content through any k must be independent of k, i.e.

$$\frac{d}{dk}\left(\frac{Gk}{\tau}\right) = 0. \tag{12}$$

Using (11),

$$GF^{\frac{1}{2}}k^{\frac{5}{2}} = \text{const.} \tag{13}$$

Using $F \sim k^{-\frac{5}{3}}$ as appropriate here, we find from (13) that

$$G = Dk^{-\frac{5}{3}}, \tag{14}$$

with D a constant, consistent with (1). This approach does not immediately give the dimensional coefficients as in (1) and (5). They can be estimated by other means. For example, we may use the expression for diffusive destruction of $\overline{\theta^2}$ spectral content,

$$\epsilon_\theta = 2\eta\int_0^\infty k^2 G\,dk. \tag{15}$$

Assume that the order of the integral is given by the part between a wave-number ${}_\theta k_L$ characterizing the large structure and the wave-number $(\epsilon/\eta^3)^{\frac{1}{4}}$ (for the cases $\nu/\eta \lesssim 1$). Then

$$\epsilon_\theta \approx 2\eta D\int_{{}_\theta k_L}^{(\epsilon/\eta^3)^{\frac{1}{4}}} k^{\frac{1}{3}}\,dk. \tag{16}$$

Integrating and neglecting ${}_\theta k_L^{\frac{4}{3}}$ relative to $(\epsilon/\eta^3)^{\frac{1}{3}}$, we get

$$D \approx \tfrac{2}{3}\epsilon_\theta\epsilon^{-\frac{1}{3}}, \tag{17}$$

consistent with (1).

Reactive mixing in the inertial-convective range

In the non-diffusive spectral range, (7) can be approximated by

$$\frac{\partial\theta_n}{\partial t} + u_i\frac{\partial\theta_n}{\partial x_i} \approx -C\theta_n. \tag{18}$$

θ_n represents any '*narrow-band*' θ-field in the convective spectral range. Multiplying by θ_n and averaging, we lose the convective term by homogeneity and are left with the simple decay equation

$$\frac{\partial\overline{\theta_n^2}}{\partial t} \approx -2C\overline{\theta_n^2}. \tag{19}$$

Integrating, we obtain the result

$$\overline{\theta_n^2} = (\overline{\theta_n^2})_0\, e^{-2Ct}. \tag{20}$$

The time rate of increase of wave-number for this reactively decaying spectral packet is still given by (10) and (11), since the first-order reaction has no effect on the geometry of the isoplethic surfaces, so

$$\frac{dk_n}{dt} = \frac{\Delta k_n}{\tau(k_n)} \approx \frac{k_n}{\tau} \approx k_n^{\frac{5}{2}} F^{\frac{1}{2}}(k_n). \tag{21}$$

With (20), we see that (12) generalizes to

$$\frac{d}{dk}\left\{\frac{Gk}{\tau} e^{2Ct}\right\} = 0, \tag{22}$$

or

$$e^{2Ct}\frac{d}{dk}\left\{\frac{Gk}{\tau}\right\} + \frac{Gk}{\tau}\frac{dt}{dk}\frac{d}{dt}\{e^{2Ct}\} = 0.$$

Using (21) for dk/dt, and (5) for $F(k)$, this integrates to

$$\boxed{G(k) \approx Bk^{-\frac{5}{3}} \exp\{3C\epsilon^{-\frac{1}{3}}k^{-\frac{2}{3}}\},} \tag{23}$$

which is shown qualitatively in figure 1. Evidently, for wave-numbers well above $k_c = C^{\frac{3}{2}}\epsilon^{-\frac{1}{2}}$, the effect of reaction on spectral shape is negligible. k_c locates the spectral region in which the rate of reactive loss is of the same order as the rate of convective spectral transfer.

The dimensional coefficient B in this generalization of (1) or (14) cannot be estimated by a condition as simple as (15) because $2\eta\int_0^\infty k^2 G\,dk$ no longer represents the total rate of destruction of $\overline{\theta^2}$. Of far narrower applicability than (15) would be estimation of D in (14) or B in (23) by

$$\overline{\theta^2} = \int_0^\infty G\,dk \approx \int_{\theta k}^{\theta k^* \text{ or } \infty} G\,dk. \tag{24}$$

Here ${}_\theta k_L$ is the 'energy-bearing' wave-number; ${}_\theta k^*$ is the end of the inertial-convective range. The limitation is, of course, that the main contribution to this integral comes from the neighbourhood of ${}_\theta k_L$, where G is unlikely to be well approximated by (14) or (23). The same limitation arises if we replace (15) by an expression for the total rate of destruction of $\overline{\theta^2}$:

$$\epsilon_\theta = 2\eta\int_0^\infty k^2 G\,dk + 2C\int_0^\infty G\,dk \tag{25}$$

$$\approx 2\eta\int_{\theta k_L}^{\theta k^*} k^2 G\,dk + 2C\int_{\theta k_L}^{\theta k^* \text{ or } \infty} G\,dk. \tag{26}$$

Having recorded these extreme doubts, but having no clearly better choice, we use (24) to estimate B. (26) is more tedious and not necessarily more reliable. Thus we write

$$\overline{\theta^2} \approx B\int_{\theta k_L}^{\theta k^*} k^{-\frac{5}{3}} \exp\{3C\epsilon^{-\frac{1}{3}}k^{-\frac{2}{3}}\}\,dk. \tag{27}$$

By integration, we get the final result

$$\overline{\theta^2} \approx -\frac{B\epsilon^{\frac{1}{3}}}{2C}\left[\exp\{3C\epsilon^{-\frac{1}{3}}{}_{\theta}k^{*-\frac{2}{3}}\} - \exp\{3C\epsilon^{-\frac{1}{3}}{}_{\theta}k_L^{-\frac{2}{3}}\}\right]. \tag{28}$$

Since ${}_{\theta}k^* \gg {}_{\theta}k_L$, we neglect the first exponential, getting the estimate

$$B \approx 2C\overline{\theta^2}\epsilon^{-\frac{1}{3}}\exp\{-3C\epsilon^{-\frac{1}{3}}{}_{\theta}k_L^{-\frac{2}{3}}\}. \tag{29}$$

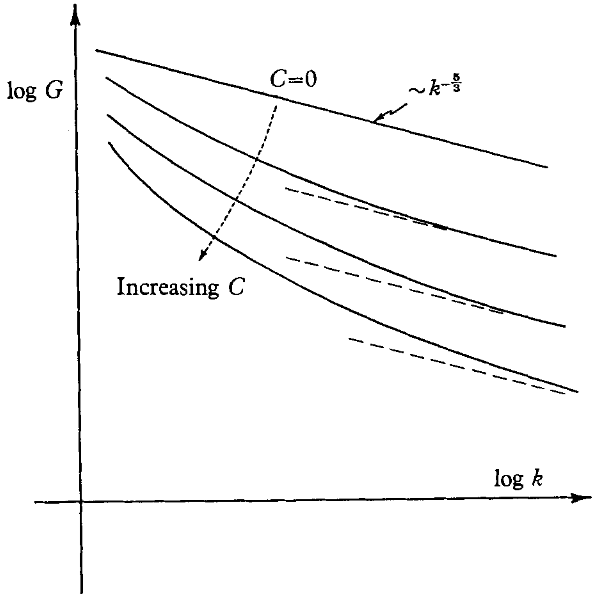

FIGURE 1. Qualitative sketch of reactant spectrum in inertial-convective range.

To gain some appreciation of (28) or (29), consider its limiting form for $C \to 0$. This is obtained most easily by putting (14) into (24), or by expanding (28) as a series, to give

$$B \to D \approx \tfrac{2}{3}\overline{\theta^2}\,{}_{\theta}k_L^{\frac{2}{3}}. \tag{30}$$

Equating this estimate to the better one [equation (17)], we find that

$$\epsilon_\theta \approx \overline{\theta^2}\,{}_{\theta}k_L^{\frac{2}{3}}\epsilon^{\frac{1}{3}}. \tag{31}$$

A similar approach to the inertial turbulent-energy spectrum gives the well-known estimate for dissipation rate in terms of large-scale parameters,

$$\epsilon \approx (\overline{u^2})^{\frac{3}{2}}\,k_L, \tag{32}$$

omitting a factor of order unity. Putting (32) into (31), we get a form looking like (32),

$$\epsilon_\theta \approx \overline{\theta^2}(\overline{u^2})^{\frac{1}{2}}\,{}_{\theta}k_L^{\frac{2}{3}}k_L^{\frac{1}{3}}. \tag{33}$$

If the large structures of velocity and contaminant fields are about equal,

$$\epsilon_\theta \approx \overline{\theta^2}(\overline{u^2})^{\frac{1}{2}}\, k_L. \tag{34}$$

In any case, we see that (29) represents the same kind of approximation as one used in obtaining (32).

Reactive mixing in the viscous-convective range and beyond

In the limiting case of mixing at very large Schmidt number, ν/η, Batchelor (1959) pointed out the possibility of a locally isotropic viscous-convective range as outlined in the Introduction. His initial arguments are unaffected by the addition of our first-order, passive chemical reaction.

This spectral range is characterized by the property that the 'smallest eddies' of the turbulent motion are much larger than the characteristic lengths of the scalar field, $(\epsilon/\nu^3)^{\frac{1}{4}} \ll k$. Therefore, if we follow a small region of fluid, the turbulent straining action can be visualized as locally uniform for a limited time. This permits analysis of the history of a Fourier element of concentration field under the joint action of turbulent straining and molecular diffusion. Inference of general spectral behaviour from a single spectral point is permissible here (unlike analysis of the turbulence field) because the diffusion equation is linear.

Batchelor chooses Cartesian axes which translate with a fluid element and are always alined with the local principal strain-rate axes. Calling these local principal strain-rates α, β, γ, equation (7) can be written as

$$\frac{\partial\theta_1}{\partial t} + \alpha x\frac{\partial\theta_1}{\partial x} + \beta y\frac{\partial\theta_1}{\partial y} + \gamma z\frac{\partial\theta_1}{\partial z} = \eta\nabla^2\theta_1 - C\theta_1. \tag{35}$$

Presumably α, β and γ can be taken constant over distances comparable with the Kolmogorov microscale $(\nu^3/\epsilon)^{\frac{1}{4}}$, and over times comparable with the Kolmogorov time $(\nu/\epsilon)^{\frac{1}{2}}$. The subscript on θ_1 reminds us that we shall analyse the history of a single Fourier element of the full θ-field.

Following Batchelor's procedure, we get instead of his equation (4.6), the equation

$$\theta_1(\mathbf{x},t) \doteqdot A_0 \exp\left\{\frac{\eta k^2}{2\gamma} - Ct\right\}\sin(kz), \tag{36}$$

where

$$k \doteqdot k_z = k_{0z}\, e^{-\gamma t} \tag{37}$$

and A_0 is initial amplitude (see equation (4.4) in Batchelor's paper). If we call $\theta_M(\mathbf{x},t)$ the field with $C = 0$, and compare with Batchelor's form for θ_M, then

$$\theta_1(\mathbf{x},t) = \theta_M(\mathbf{x},t)\, e^{-Ct}. \tag{38}$$

To use such determinate results to infer the behaviour of a random θ-field randomly strained by isotropic turbulence, we argue that (36) applies to any Fourier element whose $k \gg (\epsilon/\nu^3)^{\frac{1}{4}}$. The magnitude of the maximum negative strain-rate is taken to be the root-mean-square value of strain-rate for the turbulence, i.e. we set

$$\gamma \approx -\tfrac{1}{2}(\epsilon/\nu)^{\frac{1}{2}}. \tag{39}$$

Supposing that the θ-field is maintained statistically steady by continuous feeding into the large structure, we use (37) to estimate the kinematic history of any narrow slab of the broadband θ-spectrum as it migrates to larger and larger k under the convective straining of the turbulence. Then (36) tells us how fast this slab loses $\overline{\theta^2}$-content during any time interval of spectral travel.

The time required for a narrow slab to travel from k' to k follows from (37). It is

$$T = -\frac{1}{\gamma}\log\left(\frac{k}{k'}\right), \tag{40}$$

given by Batchelor. The change in slab width as it goes from k' to k is found to be

$$\delta k/k = \delta k'/k'. \tag{41}$$

During this time interval, the contribution to $\overline{\theta^2}$ carried by this slab decreases in the ratio given by the square of (36), with (40) for the travel time, or

$${}_{k'}\mathscr{R}_k = \left(\frac{k}{k'}\right)^{2C/\gamma} \exp\left\{\frac{\eta}{\gamma}(k^2 - k'^2)\right\}. \tag{42}$$

To compare the actual spectral levels at k' and k, we start with

$$G(k)\,\delta k = G(k')\,\delta k'\,{}_{k'}\mathscr{R}_k. \tag{43}$$

By use of (41) and (42), this leads to

$$k^{(1-2C/\gamma)}G(k)\,e^{-\eta k^2/\gamma} = k'^{(1-2C/\gamma)}G(k')\,e^{-\eta k'^2/\gamma}. \tag{44}$$

By inspection [or by using (44) we set up a difference equation, thence a differential equation which can be solved], we arrive at the spectral form

$$G(k) = Nk^{(2C/\gamma)-1}\exp\{\eta k^2/\gamma\}, \tag{45}$$

which reduces to Batchelor's mixing form for $C = 0$. $N = \text{const.}$ Introducing (39) for γ, we derive the final result

$$\boxed{G(k) = Nk^{-(1+4C\nu^{\frac{1}{2}}\epsilon^{-\frac{1}{2}})}\exp\{-2(k/k_B)^2\},} \tag{46}$$

where $k_B \equiv (\epsilon/\nu\eta^2)^{\frac{1}{4}}$. The wave-number $k_c = C^{\frac{3}{2}}\epsilon^{-\frac{1}{2}}$ does not play a direct role in the *shape* of this part of the spectrum. It does, however, affect the entire local level through N.

In the viscous-convective range, $k^* \ll k \ll k_B$, (46) simplifies to a power law,

$$G(k) \approx Nk^{-(1+4C\nu^{\frac{1}{2}}\epsilon^{-\frac{1}{2}})}. \tag{47}$$

We note parenthetically that (47) (and Batchelor's limiting form for $C = 0$) can also be obtained by application of (22) with the spectral transfer time taken as $\tau = \text{const.} = (\nu/\epsilon)^{\frac{1}{2}}$.

Qualitatively, the combination of (23) and (45) gives a θ-spectrum like figure 2.

To evaluate N, we require that (46) [hence (47)] match the lower wave-number form, equation (23), at $k = k^* \equiv (\epsilon/\nu^3)^{\frac{1}{4}}$. This gives

$$N = Bk^{*(-\frac{2}{3}+4C\nu^{\frac{1}{2}}\epsilon^{-\frac{1}{2}})}\exp\{3C\epsilon^{-\frac{1}{3}}k^{*-\frac{2}{3}}\}, \tag{48}$$

or

$$N = B(\epsilon/\nu^3)^{(-\frac{1}{6}+C\nu^{\frac{1}{2}}\epsilon^{-\frac{1}{2}})}\exp\{3C(\nu/\epsilon)^{\frac{1}{2}}\}. \tag{48a}$$

If we take (29) as a rough approximation for B, and neglect ${}_\theta k_L^{\frac{2}{3}}$ relative to $k^{*\frac{2}{3}}$ in the exponent, (48) becomes

$$N \approx 2\overline{\theta^2}C(\nu/\epsilon)^{\frac{1}{2}}\,k^{*-C\nu^{\frac{1}{2}}\epsilon^{-\frac{1}{2}}}\exp\left\{-3C\left(\frac{\nu}{\epsilon}\right)^{\frac{1}{2}}\left(\frac{k^*}{{}_\theta k_L}\right)^{\frac{2}{3}}\right\}. \tag{49}$$

It should be noted that the argument of the (constant) exponential can be written as $\{-3(k_c/{}_\theta k_L)^{\frac{2}{3}}\}$ [see remarks after equation (23)]. The ubiquity of the dimensionless reaction rate $C(\nu/\epsilon)^{\frac{1}{2}}$ is notable here, as in earlier results.

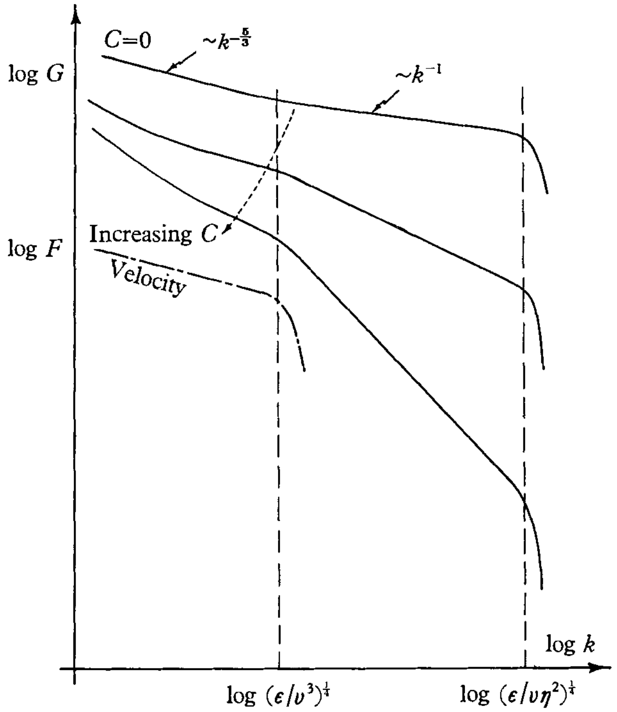

FIGURE 2. Qualitative sketch of reactant spectrum for $\nu/\eta \gg 1$.

The relation (46) reduces immediately to Batchelor's mixing case when we put $C = 0$. The corresponding simplification of the constant coefficient N follows by noting that (48) then reduces to

$$N \to Bk^{*-\frac{2}{3}}, \tag{50}$$

while

$$B \to D \approx \epsilon_\theta \epsilon^{-\frac{1}{3}}, \tag{51}$$

in accordance with (1). From (50) and (51),

$$N \to \epsilon_\theta(\nu/\epsilon)^{\frac{1}{2}}, \tag{52}$$

in essential agreement with the $C = 0$ theory.

Reactive mixing in the inertial-diffusive range

For $\nu/\eta \ll 1$, there exists an inertial-diffusive range in the θ-spectrum, its wave-number characterized by (4). For this case we can simply follow the procedure of Batchelor *et al.* (1959), with the addition of the reactive term as in equation (7).

The Fourier transform of (7) can be written as

$$\frac{\partial P(\mathbf{k},t)}{\partial t}+i\int k'_j Q_j(\mathbf{k}-\mathbf{k}',t)\,P(\mathbf{k}',t)\,d\mathbf{k}' = -\eta k^2 P(\mathbf{k},t)-CP(\mathbf{k},t). \tag{53}$$

The simplifying theoretical arguments presented by Batchelor *et al.* (1959) can be summarized and adapted as follows.

(*a*) Since the θ-spectrum doubtless falls off much more rapidly than the $\mathbf{u}$-spectrum in this range, the principal contribution to the integral must come at values of k' of the order of and less than $(\epsilon/\eta^3)^{\frac{1}{4}}$. It follows that over most of the spectral range of this analysis

$$k' \ll k \tag{54}$$

in the important part of the integral.

(*b*) The diffusion and reaction time for a Fourier component of θ around wave-number k is of order $(\eta k^2+C)^{-1}$, but the convective (quasi-source) effect has the inertial range characteristic time $(\epsilon k^2)^{-\frac{1}{3}}$ [which is the same as equation (11)]. With $k \gg (\epsilon/\eta^3)^{\frac{1}{4}}$, the latter time is much the larger. Therefore the time-derivative term can be neglected in equation (53) when it is applied to the wave-number range of interest here.

Neglect $\partial P/\partial t$ in (53), then multiplying the remainder by its complex conjugate, and continuing to follow the Batchelor, Howells & Townsend arguments by using (54) to justify neglecting the statistical connexion between the (spectrally) widely separated factors in this integrand ($|\mathbf{k}-\mathbf{k}'| \gg k'$), we finally get the following extension of their equation (8):

$$(\eta k^2+C)^2\,G(k) \approx \tfrac{2}{3}\epsilon_\theta^* F(k)/\eta, \tag{55}$$

where ϵ_θ^* is the rate of destruction of $\overline{\theta^2}$ by diffusion. Introducing (5) for F, we arrive at

$$\boxed{G(k) \approx \frac{1}{3}\,\frac{\epsilon_\theta^*\epsilon^{\frac{2}{3}}}{\eta}\,\frac{1}{k^{\frac{5}{3}}[\eta k^2+C]^2},} \tag{56}$$

which reduces properly for $C=0$. With $C \neq 0$, it approximates the pure-mixing form in the wave-number range $k \gg (C/\eta)^{\frac{1}{2}}$, which is a measure of the spectral location where reactive loss rate and diffusive loss rate are comparable.

Since (56) has no arbitrary constant coefficient (except of order unity as a consequence of the various approximations *en route*), the matching of (56) with (23) at the wave-number ${}_\theta k^* = (\epsilon/\eta^3)^{\frac{1}{4}}$ just gives a relation among a number of characteristic constants in the problem. The matching leads directly to

$$B\exp\left\{3C\sqrt{\frac{\eta}{\epsilon}}\right\} \approx \frac{1}{3}\,\frac{\epsilon_\theta^*\epsilon^{-\frac{1}{3}}}{[1+C(\eta/\epsilon)^{\frac{1}{2}}]^2}. \tag{57}$$

With (29) as an estimate for B, this can be written as

$$\epsilon_\theta^* \approx 6C\overline{\theta^2}\left[1+C\sqrt{\frac{\eta}{\epsilon}}\right]^2\exp\left\{-3C\sqrt{\frac{\eta}{\epsilon}}\left[\left(\frac{{}_\theta k^*}{{}_\theta k_L}\right)^{\frac{2}{3}}-1\right]\right\}. \tag{58}$$

This approximation is not correct for $C \to 0$. The non-reacting limit requires that the entire expression (28) be used for B.

The qualitative characteristics of reacting spectra for $\nu/\eta \ll 1$ are sketched in figure 3. We notice in (58) that for this case the appropriate dimensionless reaction rate is $C(\eta/\epsilon)^{\frac{1}{2}}$ instead of $C(\nu/\epsilon)^{\frac{1}{2}}$.

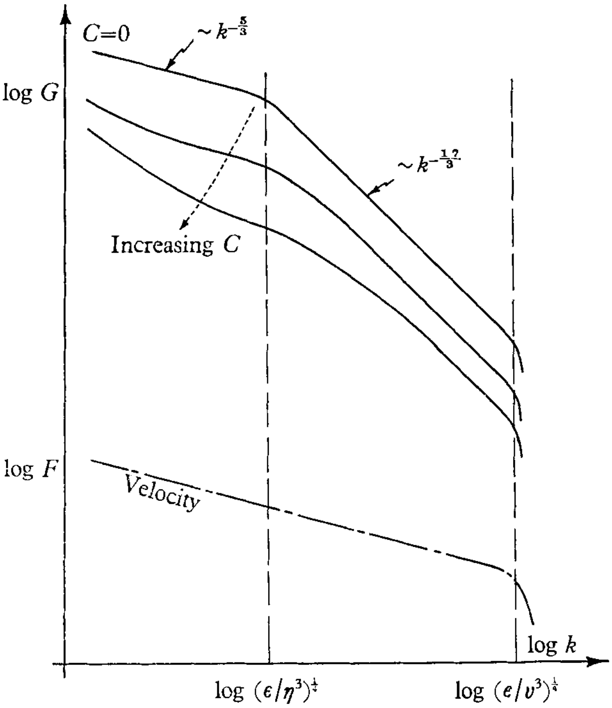

FIGURE 3. Qualitative sketch of reactant spectrum for $\nu/\eta \ll 1$.

Since (58) gives an estimate for the rate of diffusive destruction of $\overline{\theta^2}$, the total rate at which $\overline{\theta^2}$ is fed into the spectrum is just

$$\epsilon_\theta = \epsilon_\theta^* + 2C\overline{\theta^2}. \tag{59}$$

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